NOISE ON CAPACITOR COMBINATIONS

BACKGROUND

There are various situations in noise analysis where a complex of resistors, capacitors and switches result in transfer of signal to a capacitor. Each component of the circuit contributes noise, and every switch results in a partial transfer of that noise to other components. What is considered here is the question of whether the noise on an output capacitor in such a circuit will differ from the usual kTC noise on a stand-alone capacitor. We will show, using thermodynamics and simple noise analysis techniques, that the noise on a capacitor subject only to Johnson noise will always be the same, regardless of how it comes to be set.

CAPACITORS CONNECTED IN PARALLEL

Figure 1 shows the circuit to be considered. Circuit operation occurs as follows: With S_2 initially closed, S_1 is closed and then opened again, resetting the parallel combination of C_1 and C_2 to ground. Because of noise, the net charge on the parallel combination of C_1 and C_2 is not necessarily zero. Then S_2 is opened again. The charge present after the reset is now split between C_1 and C_2 in proportion to their values, and an additional component of noise is introduced to reflect the uncertainty in partitioning the charge.

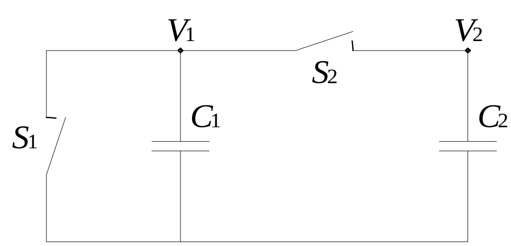

FIGURE 1 – CIRCUIT FOR ANALYSIS OF CAPACITOR NOISE.

The noise from resetting the parallel combination of capacitors is:

$$q^2 = k_B T\left(C_1 + C_2\right). \tag{1}$$

The shared portion of that reset noise on C_1 is

$$q_1^2 = q^2\left(\frac{C_1}{C_1 + C_2}\right)^2 = kT_B\frac{C_1^2}{C_1 + C_2}, \tag{2}$$

and on C_2,

$$q_2^2 = kT_B\frac{C_2^2}{C_1 + C_2}. \tag{3}$$

The noise voltage on the two capacitors is the same:

$$v_a^2 = \frac{q_1^2}{C_1^2} = \frac{q_2^2}{C_2^2} = \frac{kT_B}{C_1 + C_2}\,. \qquad (4)$$

SEPARATING THE CAPACITORS

If two capacitors having no net charge are joined together and then separated, there is a noise charge on each of them, constrained by

$$q_2 = -q_1 = q\,. \qquad (5)$$

Because there is really only one degree of freedom, the equipartition theorem of thermodynamics gives the total fluctuation energy:

$$\tfrac{1}{2}\frac{q_1^2}{C_1} + \tfrac{1}{2}\frac{q_2^2}{C_2} = \tfrac{1}{2}k_B T\,, \qquad (6)$$

so that

$$q^2 = k_B T \frac{C_1 C_2}{C_1 + C_2}\,. \qquad (7)$$

Therefore the voltage noise values across each capacitor are

$$v_{1b}^2 = \frac{q^2}{C_1^2} = \frac{C_2}{C_1 + C_2}\frac{k_B T}{C_1} \qquad (8)$$

and

$$v_{2b}^2 = \frac{Q^2}{C_2^2} = \frac{C_1}{C_1 + C_2}\frac{k_B T}{C_2}\,, \qquad (9)$$

where the *b* suffix to the subscripts distinguishes the noise of separation from the initial noise of charging, which was designated by *a*.

We can obtain the same result by examining the voltage-noise model for separating the two capacitors. Figure 2 shows the equivalent circuit for the parallel combination of two capacitors. There v is the Johnson noise voltage generated by the resistor, and v_1 and v_2 are the resulting noise voltages on C_1 and C_2, respectively. The resistor, which simulates a switch, we think of as gradually increasing in value from a "short" to an "open", at which time the capacitors are separated. As the resistance increases, the response time for fluctuations of charge between the two capacitors also increases. Phrased another way, the noise bandwidth decreases as the resistance increases. As the bandwidth approaches zero, the temporal fluctuations decrease, and the charge becomes fixed at some value.

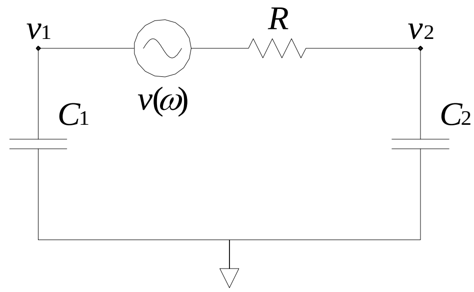

FIGURE 2 – EQUIVALENT CIRCUIT FOR COMPUTING NOISE AFTER SEPARATING TWO CAPACITORS.

Conservation of current at node 1 demands that for every frequency ω,

$$\frac{i\omega C_2}{1+i\omega RC_2}\left[v_1(\omega)+v(\omega)\right]+i\omega C_1 v_1(\omega)=0. \tag{10}$$

The disturbance at node 1 is therefore

$$v_1(\omega)=-\frac{C_2}{C_1+C_2}\frac{1}{1+i\omega RC_{\|}}v(\omega), \tag{11}$$

where

$$C_{\|}=\frac{C_1 C_2}{C_1+C_2}, \tag{12}$$

so the noise spectral density at node 1 is

$$v_1^2(\omega)=\left(\frac{C_2}{C_1+C_2}\right)^2\frac{1}{1+\omega^2R^2C_{\|}^2}v^2(\omega). \tag{13}$$

The integrated noise is

$$v_{1b}^2=\tfrac{1}{2\pi}\int_0^\infty\left|v_1(\omega)\right|^2 d\omega=\frac{C_2}{C_1+C_2}\frac{k_BT}{C_1}. \tag{14}$$

Similarly,

$$v_2(\omega)=\frac{1}{1+i\omega RC_2}\left[v_1(\omega)+v(\omega)\right]=\frac{C_1}{C_1+C_2}\frac{1}{1+i\omega RC_{\|}}v(\omega), \tag{15}$$

and

$$v_{2b}^2=\tfrac{1}{2\pi}\int_0^\infty\left|v_2(\omega)\right|^2 d\omega=\frac{C_1}{C_1+C_2}\frac{k_BT}{C_2}, \tag{16}$$

confirming our earlier analysis.

This noise does not take into account that the charge being shared by C_1 and C_2 may differ from its nominal value, as shown in Section 0 above. That is, the initial charge shared by the two capacitors my randomly differ from zero – per Section 1 – and the noise components labeled *b* are not correlated with those labeled *a*.

THE NET RESULT

Now, if the parallel combination of C_1 and C_2 is reset, and the two capacitors are subsequently separated, the net noise on C_1 is given by adding Eq. (4) and Eq. (11), resulting in

$$\delta v_1^2=\delta v_a^2+\delta v_{1b}^2=\frac{k_BT}{C_1+C_2}\left(1+\frac{C_2}{C_1}\right)=\frac{k_BT}{C_1}. \tag{17}$$

Similarly,

$$\delta v_2^2=\frac{k_BT}{C_2}. \tag{18}$$

Thus, the net result is the same as if C_1 and C_2 had been reset separately.

COMBINING CAPACITORS

Consider now the case in which the two capacitors are initially reset separately. We have

$$v_{1a}^2 = \frac{k_B T}{C_1} \qquad \text{and} \qquad v_{2a}^2 = \frac{k_B T}{C_2}. \tag{19}$$

If now they are electrically connected, the charge on the two capacitors is shared between them, and the total variance is

$$q^2 = C_1^2\, v_1^2 + C_2^2\, v_2^2 = k_B T\left(C_1 + C_2\right), \tag{20}$$

and, as expected,

$$v^2 = \frac{q^2}{\left(C_1 + C_2\right)^2} = \frac{k_B T}{C_1 + C_2}. \tag{21}$$

CONCLUSION

A capacitor or combination of capacitors, if subject only to Johnson noise, has the same total noise regardless of how the voltage on the capacitor comes to be set. Specifically, this means that the noise of a switched-capacitor fed by a resistive source will be only the kTC noise on the final capacitor.