NON-EQUILIBRIUM JOHNSON NOISE

BACKGROUND

It is well know that when a capacitor is set to a fixed voltage, the voltage on the capacitor after the voltage is removed is not exactly equal to the reference voltage. Rather, there is a noise associated with the process, and the noise, the standard deviation of the voltage across the capacitor, is

$$v_{rst} = \sqrt{\frac{k_B T}{C}}\,, \tag{1}$$

where k_B is Boltzmann's constant and T is the temperature of the capacitor C. This can be demonstrated thermodynamically by recognizing that the fluctuation has only one degree of freedom, and the energy of the fluctuation is

$$\tfrac{1}{2} C\, v_{rst}^2 = \tfrac{1}{2} k_B T\,. \tag{2}$$

If the capacitor is subsequently connected to another circuit with its noise source, the value established by the reset may or may not affect the eventual result. Two examples illustrate this point. If the second noise source were simply another reset operation, then the second sample would be totally independent of the first, although the noise would be the same. On the other hand, if the second noise source were an infinite impedance (*i.e.*, the switch remained open indefinitely), then the initial value would be sustained.

In the following treatment we examine three practical cases: (1) the second circuit is a resistor connected to a voltage source; (2) the second circuit is a constant-current source; and (3) the second circuit is a resistor connected to voltage source, and the capacitor is in the feedback loop of a high-gain inverting amplifier.

RESISTOR AND VOLTAGE SOURCE

If the capacitor is subsequently connected to another voltage source through a resistor, the voltage on the capacitor will eventually equilibrate with the voltage source, and the noise on the capacitor will be the Johnson noise from the resistor integrated through a bandwidth set by the resistor-capacitor filter, which is the same is in Eq. (1) above. This is demonstrated by

$$v_J^2 = \frac{1}{2\pi}\int_0^\infty \frac{4k_B TR}{1+\omega^2 R^2 C^2}\, d\omega = \frac{k_B T}{C}. \tag{3}$$

Although the noise values are the same, it is clear that after sufficient time has passed the initial reset value with its noise has no bearing on the final value. The question is this: How does one value of the $k_B T/C$ noise evolve into the other; *i.e.*, what is the noise at intermediate times?

In Figure 1 below, the switch S is initially closed and then opened again. At time $t = 0$, the instant of opening S, the charge on the capacitor C is nominally zero. Because of noise sources, however, any given instance of this operation will leave a residual charge on C, and so $V(t = 0)$ will be non-zero. Each instance will yield a different initial charge, but that charge will set the initial condition for the evolution of $V(t)$ until equilibrium is achieved.

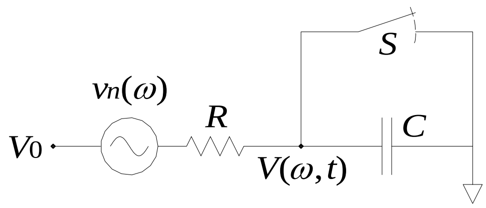

FIGURE 1 – EQUIVALENT CIRCUIT FOR NOISE ANALYSIS FOR A CAPACITOR THAT IS RESET AND THEN CHARGED THROUGH A RESISTOR.

The initial condition is

$$V(t=0)=v_0, \tag{4}$$

where the value v_0 results from noise in the reset operation. The rms value of v_0 over many such experiments is given by Eq. (1). The equation of motion describing current flowing into C is

$$I(\omega,t)=\frac{V_0+v_n(\omega)e^{i\omega t}-V(\omega,t)}{R}, \tag{5}$$

where $I(\omega,t)$ is the current at time t, V_0 is the bias voltage applied to the resistor, $v_n(\omega)$ is the amplitude of the noise source voltage at frequency ω (*i.e.*, the noise power spectral density), and R is the value of the resistor.

The charge on C at any time is

$$Q(\omega,t)=CV(\omega,t). \tag{6}$$

Differentiating Eq. (6) and combining it with Eq. (5) leads to

$$I(\omega,t) = \dot{Q}(\omega,t) = C\dot{V}(\omega,t) = \frac{V_0+v_n(\omega)e^{i\omega t}-V(\omega,t)}{R}, \tag{7}$$

or

$$\tau\frac{dV(\omega,t)}{dt}+V(\omega,t) = V_0+v_n(\omega)e^{i\omega t}, \tag{8}$$

where

$$\tau=RC. \tag{9}$$

The solution is

$$V(\omega,t) = V_0\left(1-e^{-t/\tau}\right)+v_0e^{-t/\tau}+\frac{v_n(\omega)}{1+i\omega\tau}\left(e^{i\omega t}-e^{-t/\tau}\right). \tag{10}$$

The variance is defined by

$$v^2(\omega,t)=\left\langle V^2(\omega,t)\right\rangle-\left\langle V(\omega,t)\right\rangle^2, \tag{11}$$

where the symbols $\langle x(\omega,t)\rangle$ represent averages in the following sense: Imagine a sequence (ensemble) of experiments in which the capacitor is reset, and then the voltage on C is measured after a time t has elapsed. In each experiment exactly the same time elapses as in every other experiment. The average, then, is computed over the sequence of experiments for a specified elapsed time. It is specifically *not* a time average.

Expanding Eq. (11) using Eq. (10),

$$v^2(\omega,t)=\left\langle V_0^2\left(1-e^{-t/\tau}\right)^2+v_0^2\,e^{-2t/\tau}+\frac{v_n^2(\omega)}{1+\omega^2\tau^2}\left[1-2\cos(\omega t)e^{-t/\tau}+e^{-2t/\tau}\right]\right\rangle-V_0^2\left(1-e^{-t/\tau}\right)^2, \quad (12)$$

where we have used the fact that the ensemble averages of v_0 and v_n are zero, as also is the ensemble average of their product (because they are not correlated); *i.e.*,

$$\langle v_n\rangle=\langle v_0\rangle=\langle v_0 v_n\rangle=0. \quad (13)$$

Continuing,

$$v^2(\omega,t)=v_0^2e^{-2t/\tau}+\frac{v_n^2(\omega)}{1+\omega^2\tau^2}\left(1-2\cos(\omega t)e^{-t/\tau}+e^{-2t/\tau}\right). \quad (14)$$

When we integrate over all frequencies, considering that the different frequency modes are independent, we get[1]

$$v^2(t)=v_0^2\,e^{-2t/\tau}+\frac{v_n^2}{4\tau}\left(1-e^{-2t/\tau}\right), \quad (15)$$

where we have assumed that v_n is independent of ω; *i.e.*, that the noise is white.

From this it is evident that the initial reset noise decays with time, and the Johnson noise from the resistor grows with time. This first point is intuitive because, without the Johnson noise, any initial charge on C will not affect the ultimate charge as $t\rightarrow\infty$. The second point is also intuitive, because the charge on C cannot change instantaneously. Even the exponential decay and growth of the two noise sources is intuitive. Note that the implication of Eq. (15) is that if the initial noise and the final noise are equal, then the noise is constant in time. This, of course, does not mean that the deviation of the actual charge on C from the nominal time-dependent charge is constant, but only that the rms value of the deviation is the same at every instant. Particularly, if

$$\langle v_0^2\rangle=\frac{k_BT}{C} \quad (16)$$

and

$$\langle v_n^2\rangle=4k_BTR, \quad (17)$$

then

$$\langle v^2(t)\rangle=\frac{k_BT}{C} \quad (18)$$

at all times.

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[1] I.S. Gradshteyn and I.M. Ryzhik, **Table of Integrals, Series, and Products**, Academic Press, 1980, Formula 3.723-2:
$$\int_0^\infty \frac{\cos(\alpha x)}{\beta^2+x^2}dx=\frac{\pi}{2\beta}e^{-\alpha\beta}\quad \text{for}\ \ \alpha\ge 0,\ \Re(\beta)>0.$$

It is interesting to note that the second term in Eq. (15) looks like an integrator if $t << \tau$:

$$\frac{v_n^2}{4\tau}\left(1-e^{-2t/\tau}\right) \approx \frac{v_n^2}{R^2}\frac{t^2}{C^2}\frac{1}{2t}. \tag{19}$$

Here the R^2 in the denominator converts the noise voltage to a current, the factor t^2/C^2 integrates the charge onto the capacitor, and the factor $1\,/\,2t$ is the bandwidth for an integrator. This does not mean that this integrated noise adds to the initial k_BT/C noise, because the decay of the latter in the small t approximation combines with Eq. (19) to exactly maintain the validity of Eq. (18).

CONSTANT-CURRENT SOURCE

Figure 2 shows a similar noise problem, except that the capacitor, after being reset, is charged by a current source rather than through a resistor. At first glance, this might appear to be the same as charging through a very large resistor. There is some truth to that, but not if charging continues to the point of steady state (which, in principle, never really happens with a true current source). If charging is only for a fixed time, then a large R means that the condition $t << \tau$ holds for a longer period, and so the resistor will look like a current source for a longer period. However, the condition Eq. (18) still holds true at every instant. The difference between a large resistor and a true current source is that for a true current source the instantaneous value of $V(t)$ does not affect the instantaneous current. In the case of charging through a resistor, a fluctuation in $V(t)$ changes the voltage across the resistor, and so the current changes in a manner that tends to compensate for the fluctuation. For example, if noise causes $V(t)$ to drop below its nominal value, then the voltage drop across the resistor is greater than its nominal value. This increased voltage causes more current to flow, so that $V(t)$ tends back toward its nominal value. For a true current source the current is constant regardless of the state of the load (within limits, of course). Another way of understanding this is that with a voltage source the voltage on the capacitor modifies the charging rate of the capacitor, thereby limiting the noise bandwidth, but for a current source the restoring force has no bearing on the rate of charging. For this reason, the initial noise on C does not dissipate as the capacitor is charged.

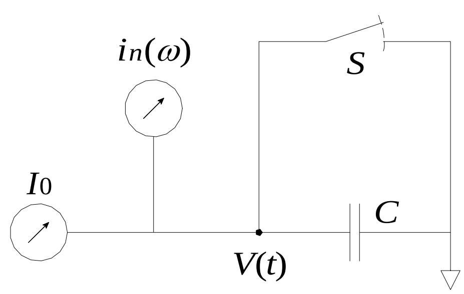

FIGURE 2 – EQUIVALENT CIRCUIT FOR NOISE ANALYSIS FOR A CAPACITOR THAT IS RESET AND THEN CHARGED WITH A CURRENT SOURCE.

The equation that appropriately describes Figure 2 is

$$V(t) = v_0 + I_0\frac{t}{C} + \frac{i_n(\omega)}{C}\int_0^t e^{i\omega t'}dt' = v_0 + I_0\frac{t}{C} + i_n(\omega)\,\frac{t}{C}\,e^{i\frac{\omega t}{2}}\,\mathrm{sinc}\left(\frac{\omega t}{2}\right). \tag{20}$$

Applying Eq. (11) again, the result is

$$v^2 = v_0^2 + i_n^2(\omega)\frac{t^2}{C^2}\text{sinc}^2\left(\frac{\omega t}{2}\right). \tag{21}$$

Integrating over all frequencies as before,

$$v^2 = v_0^2 + i_n^2\frac{t}{2C^2} = v_0^2 + \left(i_n^2\frac{1}{2t}\right)\frac{t^2}{C^2}. \tag{22}$$

The factor $1/2t$ is the noise bandwidth for an integrator, so the term in parentheses is the net variance of the current. The t^2 factor converts current variance to charge variance, and dividing by C^2 converts that to voltage variance.

If the noise in the current source is Johnson noise, then

$$v^2 = \frac{k_B T}{C}\left(1 + \frac{2t}{\tau}\right). \tag{23}$$

The total noise is therefore the sum of the reset noise ($k_B T/C$) and Johnson noise with the bandwidth set by an integrator.

INTEGRATION ON A FEEDBACK CAPACITOR

A third realistic alternative is shown in Figure 3. Here the source is an ordinary resistor, but neither side of the integration capacitor is grounded. The signal is not read from the capacitor integration point, but from the amplifier output. When switch *S* is closed and reopened, the noise at the amplifier output is not as obvious as it was in the previous cases.

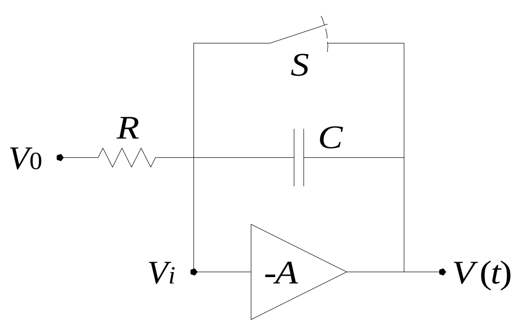

FIGURE 3 – CIRCUIT FOR INTEGRATION OF CURRENT ON A FEEDBACK CAPACITOR ACROSS A HIGH-GAIN INVERTING AMPLIFIER.

To better understand the interactions of the various components, we will use the equivalent circuit of Figure 4. Here we have substituted a resistor R_s for the switch S. When the switch is closed we have $R_s \ll R$, and the when the switch is open we have $R_s \gg R$.

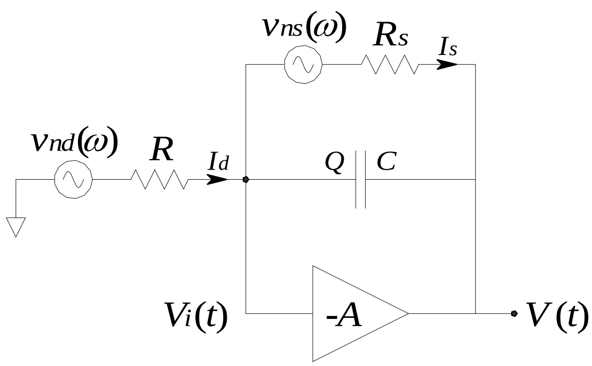

FIGURE 4 – EQUIVALENT CIRCUIT FOR NOISE ANALYSIS OF INTEGRATION OF CURRENT ON A FEEDBACK CAPACITOR ACROSS A HIGH-GAIN INVERTING AMPLIFIER.

The equations governing operation of the circuit in Figure 4 are

$$V_i(t) + v_{ns} - V(t) = I_s R_s \tag{24}$$

$$Q = C\left[V_i(t) - V(t)\right] \tag{25}$$

$$V(t) = -A V_i(t) \tag{26}$$

$$v_{nd} - V_i(t) = I_d R \tag{27}$$

and
$$I_d = I_s + \dot{Q}. \tag{28}$$

These can be combined to give

$$(A+1)C\dot{V}(t) + \left(\frac{A+1}{R_s} + \frac{1}{R}\right)V(t) = A\left(\frac{v_{ns}}{R_s} - \frac{v_{nd}}{R}\right)e^{i\omega t}, \tag{29}$$

where
$$v_{ns} = 4k_B T R_s \tag{30}$$

and
$$v_{nd} = 4k_B T R, \tag{31}$$

and these two noise sources are uncorrelated. The solution is

$$V(t) = v_0 e^{-t/\tau} + A\frac{R'}{R_s}\frac{v_{ns}}{1+i\omega\tau}\left(e^{i\omega t} - e^{-t/\tau}\right) - A\frac{R'}{R}\frac{v_{nd}}{1+i\omega\tau}\left(e^{i\omega t} - e^{-t/\tau}\right), \tag{32}$$

where
$$R' = \left(\frac{A+1}{R_s} + \frac{1}{R}\right)^{-1}, \tag{33}$$

and
$$C' = (A+1)C. \tag{34}$$

and
$$\tau = R'C' = \frac{(A+1)RR_s}{(A+1)R + R_s}C. \tag{35}$$

From this, using Eq. (11), we find the noise at the amplifier output to be

$$v^2(t) = v_0^2 e^{-2t/\tau} + \left(\frac{A}{A+1}\right)^2 \frac{R+R_s}{R+\frac{R_s}{A+1}} \frac{k_B T}{C}\left(1-e^{-2t/\tau}\right). \tag{36}$$

If we wait long enough,

$$v^2(t) \xrightarrow{t\to\infty} \left(\frac{A}{A+1}\right)^2 \frac{R+R_s}{R+\frac{R_s}{A+1}} \frac{k_B T}{C}. \tag{37}$$

We might be tempted to simply insert $k_B T/C$ for v_0, but the presence of the amplifier confuses the situation somewhat. There are two rational and conflicting approaches to determining v_0. First, we could note that it is the voltage *across* the capacitor that should fluctuate by $k_B T/C$, and since the capacitor is placed across the amplifier we should have

$$\overline{v_{out} - v_{in}}^2 = \frac{k_B T}{C}. \tag{38}$$

Replacing v_{in} with $(-v_{out})/A$, and considering that the input and output noises are correlated, produces

$$v_{out}^2 = \left(\frac{A}{A+1}\right)^2 \frac{k_B T}{C} \approx \frac{k_B T}{C}. \tag{39}$$

On the other hand, we could consider that the noise on the capacitor should be the integrated Johnson noise produced by resistor R. Looking into the input of the amplifier, the capacitor appears to have the value $(A+1)C$, so the noise at the input should be

$$v_{in}^2 = \frac{1}{A+1}\frac{k_B T}{C}, \tag{40}$$

and at the output,

$$v_{out}^2 = \frac{A^2}{A+1}\frac{k_B T}{C} \approx A\frac{k_B T}{C}. \tag{41}$$

To resolve this dilemma we consider what actually goes on during the reset operation. While the switch S (in Figure 3) is closed, $R_s << R$, so

$$\tau \xrightarrow{R_s \ll R} R_s C. \tag{42}$$

the v_0 term decays quickly and, $v(t)$ becomes, from Eq. (36),

$$v^2(t) \xrightarrow{R_s \ll R} \left(\frac{A}{A+1}\right)^2 \frac{k_B T}{C}. \tag{43}$$

If, now, we suddenly open the switch, we capture this value.

If, on the other hand, we open the switch by slowly increasing R_s, then

$$\tau \xrightarrow{R_s \gg R} (A+1)RC. \tag{44}$$

and Eq. (36) becomes

$$v^2(t) \xrightarrow{R_s \gg R} \frac{A^2}{A+1}\frac{k_B T}{C}, \tag{45}$$

which is much greater than Eq. (43). This would be the noise if we were to open the switch and allow the circuit to achieve steady state, assuming sufficient voltage range of the amplifier output. This, of course, is an unreasonable approach to resetting the capacitor, and Eq. (43) provides the correct value for v_0 in Eq. (36).

We are interested in the case in which the switch is closed and allowed to equilibrate, and then the switch is opened. The result is Eq. (36) with v_0 given by Eq. (43), and after the switch is opened $R_s \to \infty$:

$$v^2 = \left(\frac{A}{A+1}\right)^2 \frac{k_B T}{C} e^{-2t/RC'} + \frac{A^2}{A+1}\frac{k_B T}{C}\left(1-e^{-2t/RC'}\right) = \left(\frac{A}{A+1}\right)^2 \left\{\frac{k_B T}{C} + \frac{A}{A+1}\left(\frac{4k_B T}{R}\frac{1}{2t}\frac{t^2}{C^2}\right)\right\}. \tag{46}$$

The first term shows the decay of the reset noise, and the second term shows the growth of the integrated noise. For short integration times the integrated noise is

$$v^2 = \left(\frac{A}{A+1}\right)^2 \frac{4k_B T}{R}\frac{1}{2t}\frac{t^2}{C^2}, \tag{47}$$

which is essentially the same as Eq. (19). We can break this equation down to understand it more readily. The term $4k_BT/R$ is the current noise spectral density, and $1/2t$ is the noise bandwidth for an integrator integrating for time t. The product of the noise is the current variance, which is the square of the rms noise current. Multiplying by t^2 converts to the integrated charge variance, and dividing by C^2 converts to voltage variance, which is the square of the rms noise voltage.

We can rewrite Eq. (46) to show the relative weightings of the two terms,

$$v^2 = \left(\frac{A}{A+1}\right)^2 \left[1 + A\left(1 - e^{-\frac{2t}{(A+1)\tau}}\right)\right]\frac{k_B T}{C}. \tag{48}$$

For typically short integration times,

$$v^2 \approx \left(1 + \frac{2t}{\tau}\right)\frac{k_B T}{C}. \tag{49}$$

since A>>1. This is essentially the same result as Eq. (23) for a constant-current source and an ideal integrator. That is, there is a kT/C noise from the reset, and there is an additional contribution of the integrated Johnson noise.

It is interesting to note that if the reset were to take place in such a manner that Eq. (45) would be valid, then Eq. (36), with R_s>>R, suggests that the rms noise, although large, would be constant in time.

SUMMARY

A reset capacitor has a noise associated with the reset operation, the well-known k_BT/C noise. The reset voltage determines the nominal value after the reset operation, but the standard deviation is always given by Eq. (1). If the capacitor is subsequently connected to a voltage source through a resistor, the nominal voltage on the capacitor evolves according to well-known principles. However, the noise (*i.e.*, the standard deviation) is always the same at every instant. The noise charge from the reset operation eventually dissipates, but noise from the charging resistor increases at the same rate.

If instead of a voltage source and a resistor, the capacitor is connected to a current source after being reset, then any noise that accumulates during the charging operation adds in quadrature with that of the reset operation.

If a capacitor is connected to a voltage source through a resistor, but the capacitor is in the feedback loop of a high-gain inverting amplifier, the result is essentially the same as for a constant-current source and an integrator.